Shape of Ideas : Problem Set 4

Mathematics Club IITM

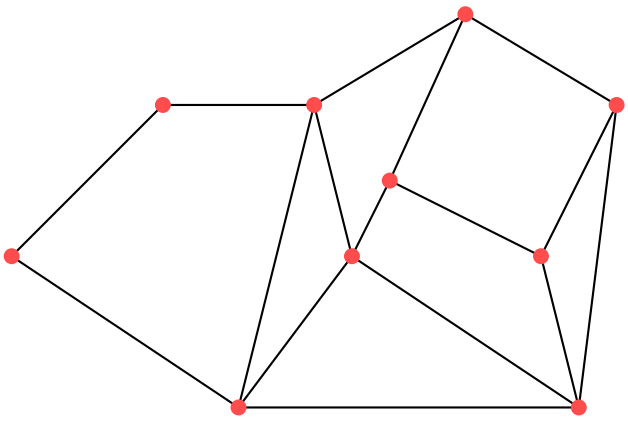
Ananth Madhav V.

Instructions

- This problem set contains **9 questions** worth **30 points**.
- **All questions are compulsory**. Even if you're unsure of the answer, write your initial thoughts, approach, or reasoning. Do not leave any question blank.
- You are encouraged to think independently and refer to credible resources if needed.
- Use of **Large Language Models** (**LLMs**) like **ChatGPT**, **Gemini**, or **Copilot** is **strictly prohibited**. If detected, the submission will be **disqualified**.
- Submit your answers in a **single PDF file**. Handwritten work is allowed but must be neatly scanned or photographed and compiled.
- Name your file as: **<your_name>_Problem_Set_4.pdf**. Example: Ananth_Problem_Set_4.pdf.
- Provide clear explanations. For theoretical questions, justify your answers. For calculations, include brief reasoning or method.
- Ensure the work is your own. Discussions are permitted, but **copying is not**. Any plagiarism will lead to **rejection**.
- Submit your assignment by **23 July 2025, EOD (End Of Day)**. Late submissions will **not be accepted** without prior approval.
- Submission is through a **Google Form**. In the form, you must **upload the file directly to the form**. Make sure the file has appropriate viewing permissions.
- Feel free to reach out to us for doubts! Contact information of the problem set creator:
  Ananth Madhav: +91 88384 53586

Questions

1. (***2 points***) For the below graph, find the:
   (a) (*1 point*) edge connectivity
   (b) (*1 point*) vertex connectivity



2. (***3 points***) Let G be a connected undirected graph with exactly four vertices of odd degree.
   (a) (*2 points*) Does an Eulerian circuit exist in G? If so, give an example. If not, why?
   (b) (*1 points*) Is it always possible to partition the edge set of such a graph into exactly two edge-disjoint trails? If yes, give an example.

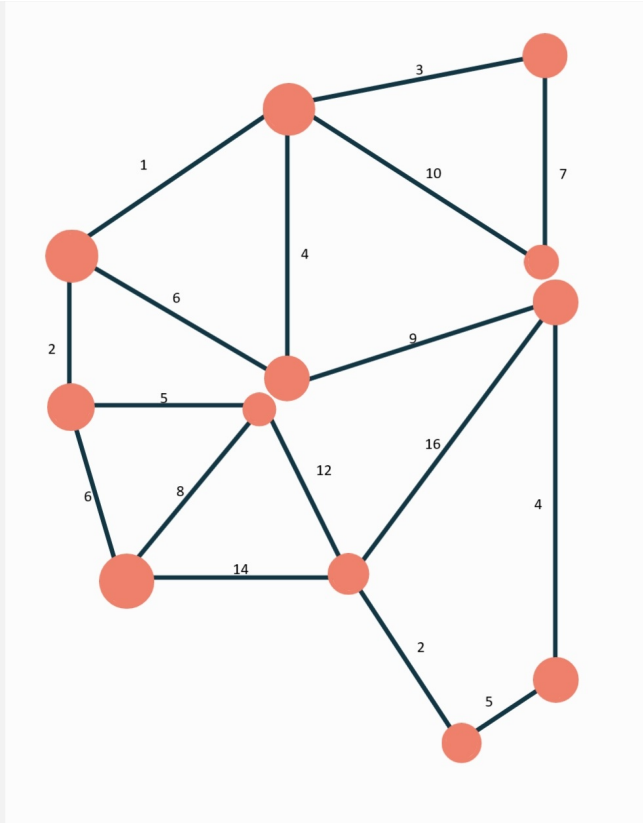
3. (***3 points***) Given a simple undirected connected graph G with n vertices, suppose that for every pair of non-adjacent vertices u and v, $\deg(u) + \deg(v) \geq n$. Prove that G contains a Hamiltonian cycle.

4. (***3 points***) Consider a bipartite graph H with bipartition (A, B), $|A| = |B| = n$, and suppose every vertex has degree at least $n/2$. Does H always contain
   (i) (*2 points*) a Hamiltonian path?
   (ii) (*1 point*) a Hamiltonian cycle?

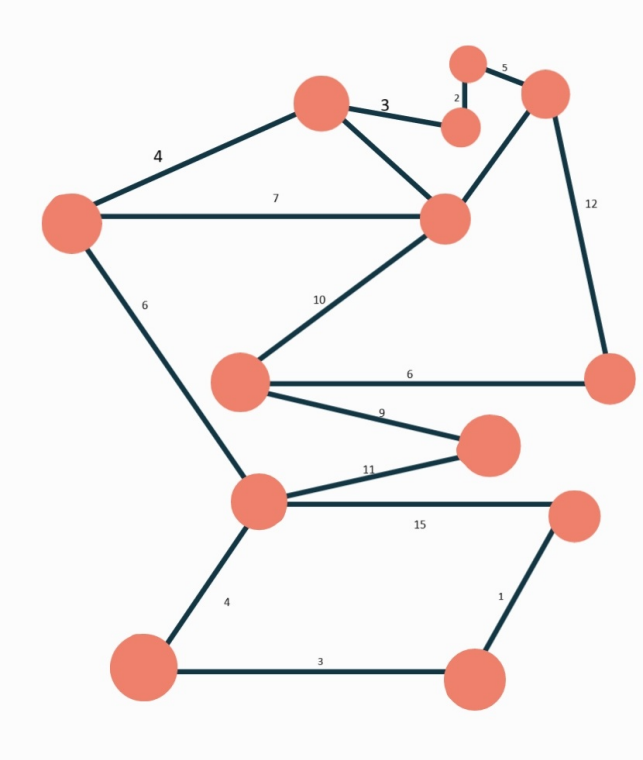
   Justify your answer.

5. (***5 points***) Let $n \geq 4$ be an integer, and let T be a labeled tree on the vertex set $1, 2, ..., n$.
   (a) (*3 points*) Let $d \geq 2$. How many labeled trees are there on n vertices in which a fixed vertex, say vertex 1, has degree exactly d? Provide an explicit formula in terms of n and d.
   (b) (*2 points*) For $n = 7$, how many labeled trees are there in which both vertex 1 and vertex 2 have degree 3?

6. (***4 points***) Find the minimum spanning tree of the following graphs, using one of the major algorithms. Explain the steps clearly.
   (a) (*2 points*)



(b) (*2 points*)



7. (***2 points***) Determine the critical 3-chromatic graphs. Give your reasoning for your answer.

8. (***5 points***) Prove the recursive bound of Ramsey numbers i.e. $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

9. (***3 points***) How does Euler's formula constrain the possible structures of planar graphs, and what does it reveal about the maximum number of edges such graphs can have in relation to their number of vertices?